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Locking-free DSP Element for Linear Elasticity Problem*

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Abstract: A four-parameter quadrilateral nonconforming finite element with DSP (double set parameters) is proposed to approximate the pure displacement linear elasticity problem for homogeneous isotropic material. It is shown that the performance of the scheme does not deteriorate as the material becomes nearly incompressible. Numerical results are also given to verify the theoretical analysis. Furthermore, the superconvergence result is obtained through a postprocessing operator. All the results can be extended to three dimensional case.

Keywords: quadrilateral nonconforming element; double set parameters; locking-free; linear elasticity; interpolation postprocessing

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1 Introduction

There now exists a large number of publications devoted to the task of constructing and analyzing finite element approximations for problems in solid mechanics, in which it is necessary to circumvent volumetric locking. Interested readers can refer to [1-4].

In order to avoid the locking effects of the nearly incompressible linear elasticity, a new nonconforming scheme is presented in this paper. And the nonconforming finite element (named as QB element) is constructed by the DSP (double set parameters) method. Then we investigate the quadrilateral QB nonconforming scheme for the linear elasticity problem. For rectangular meshes, we can equate the nonconforming method with a mixed method directly. As to the equivalent mixed formulation, we find that it exhibits some features of the well-known Q_1-P_0 element for the Stokes problem. Following the ideas presented in [5-8], and by using some special relations between QB element and Q_1 element, the Babuška-Brezzi condition[6] with a constant (inf-sup constant) independent of h is derived after filtering out some "local spurious" pressure modes. Then an optimal convergence rate is established for both displacement and smoothed pressure. Based on the mathematical analysis of the $QB-P_0$ scheme for the mixed formulation, and by introducing another modification of the mixed form as an auxiliary problem, the optimal error estimate in energy norm is obtained. Finally, using the interpolation postprocessing technique, the superconvergence relult is obtained.

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2 Notations and preliminaries

In the context of elasticity, vector- and tensor- or matrix-valued functions are written in boldface form. Let $\Omega \subset R^2$ be a convex polygonal domain. We use standard definitions for the Sobolev spaces $H^s(\Omega)$ and their associated norms $\|\cdot\|_{s,\Omega}$, and seminorms $|\cdot|_{s,\Omega}$, and

$$L_0^2(\Omega) = \Big\{q \in L^2(\Omega); \int_\Omega q\mathrm{d}x = 0\Big\}.$$

Finally, $P_k(\Omega)$ denotes the space of all polynomials on Ω of degree $\leq k$, and $Q_k(\Omega)$ denotes the space of all polynomials on Ω of degree $\leq k$ in each variable.

The pure displacement planar elasticity problem for homogeneous isotropic material can be described as

$$\begin{cases} -\mu\Delta\mathbf{u} - (\mu+\lambda)\mathrm{graddiv}\,\mathbf{u} = \mathbf{f}, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\mathbf{u}$ denotes the displacements, $\mathbf{f}$ the body forces, μ and λ are the Lamé parameters.

The corresponding variational problem is

$$\begin{cases} \text{Find } \ \mathbf{u} \in \mathbf{H}_0^1(\Omega), \ \text{ such that} \\ \mu(\nabla\mathbf{u}, \nabla\mathbf{v}) + (\lambda+\mu)(\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega). \end{cases} \tag{2}$$

The following theorem yields solvability of problem (2), which can be found in [2,9].

Theorem 2.1 Let $\Omega \subset R^2$ be a bounded convex polygonal domain, $\mathbf{f} \in \mathbf{L}^2(\Omega)$, then the variational form (2) has a unique solution $\mathbf{u} \in \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega)$, and there exists a positive constant C such that

$$\|\mathbf{u}\|_{2,\Omega} + \lambda\|\mathrm{div}\mathbf{u}\|_{1,\Omega} \leq C\|\mathbf{f}\|_{0,\Omega}. \tag{3}$$

If we define $p = (\lambda+\mu)\mathrm{div}\mathbf{u}$, then the equations of elasticity may also be written in a mixed form: find $(\mathbf{u}, p) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{cases} \mu(\nabla\mathbf{u}, \nabla\mathbf{v}) + (\mathrm{div}\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}), & \forall\, \mathbf{v} \in \mathbf{H}_0^1(\Omega), \\ (\lambda+\mu)^{-1}(p, q) - (\mathrm{div}\mathbf{u}, q) = 0, & \forall\, q \in L_0^2(\Omega). \end{cases} \tag{4}$$

This formulation is valid even in the incompressible limit $\lambda = \infty$. For the application in fluid flow, μ is the viscosity, and $\mathbf{u}$ is the velocity. At the incompressible material limit $\lambda = \infty$, we obtain from problem (4) the equation for incompressible elasticity, the same as the Stokes equation.

3 Double set parameter method and QB element

As is known that the main disadvantage of nonconforming elements is more total degrees of freedom involved than conforming ones. Taking for the rotated Q_1 element[10] as an example, its total degree of freedom is almost double as that of conforming bilinear element. A distinct idea is to combine the two elements together, and then the double set parameter method[6] follows it. The main advantage of this new approach is the nodal parameters and the degree

of freedom may be chosen independently. In principle, the nodal parameters are chosen to be simple so that the total number of unknowns in the resulting discrete system is small, while the degrees of freedom are selected to meet convergence requirement setting by the generalized patch test[11]. Then, following [12], we will present a nonconforming element, which is named QB element.

Assume that $\widehat{K}=[-1,1]\times[-1,1]$ is the reference element with vertices

$$\widehat{a}_1=(-1,-1),\quad \widehat{a}_2=(1,-1),\quad \widehat{a}_3=(1,1),\quad \widehat{a}_4=(-1,1),$$

and sides

$$\widehat{l}_1=\widehat{a}_1\widehat{a}_2,\quad \widehat{l}_2=\widehat{a}_2\widehat{a}_3,\quad \widehat{l}_3=\widehat{a}_3\widehat{a}_4,\quad \widehat{l}_4=\widehat{a}_4\widehat{a}_1.$$

The first set of degrees of freedom on $\widehat{K}$ is taken as the mean-value-oriented rotated Q_1 element

$$D(\widehat{v})=\left(\widehat{v}_{12},\widehat{v}_{23},\widehat{v}_{34},\widehat{v}_{41}\right)^{\top}, \tag{5}$$

where

$$\widehat{v}_{i(i+1)}=\frac{1}{|\widehat{l}_i|}\int_{\widehat{l}_i}\widehat{v}\mathrm{d}\widehat{s},\quad i=1,2,3,4.$$

The shape function space on reference $\widehat{K}$ is taken as

$$\widehat{P}=P_1(\widehat{K})\cup\{\xi^2-\eta^2\}. \tag{6}$$

Suppose

$$\widehat{v}=\beta_1+\beta_2\xi+\beta_3\eta+\beta_4(\xi^2-\eta^2). \tag{7}$$

Substitute (7) into (5) one can get

$$\begin{cases} \beta_1=\frac{1}{4}(\widehat{v}_{12}+\widehat{v}_{23}+\widehat{v}_{34}+\widehat{v}_{41}), & \beta_2=\frac{1}{2}(\widehat{v}_{23}-\widehat{v}_{41}), \\ \beta_3=\frac{1}{2}(\widehat{v}_{34}-\widehat{v}_{12}), & \beta_4=\frac{3}{8}(-\widehat{v}_{12}+\widehat{v}_{23}-\widehat{v}_{34}+\widehat{v}_{41}). \end{cases} \tag{8}$$

Using the double set parameters method, we take another set of nodal parameters

$$Q(\widehat{v})=\left(\widehat{v}_1,\widehat{v}_2,\widehat{v}_3,\widehat{v}_4\right)^{\top}, \tag{9}$$

i.e., the function values at 4 nodes, which are the real degrees of freedom.

Approximating the degrees of freedom $D(\widehat{v})$ by the trapezoidal rule as follows

$$\widehat{v}_{i(i+1)}=\frac{1}{2}(\widehat{v}_i+\widehat{v}_{i+1})+\widehat{\delta}_i(\widehat{v}),\quad i=1,2,3,4. \tag{10}$$

For any $\widehat{v}\in P_1(\widehat{K})$, these discretizations are exact, and an application of the Bramble-Hilbert lemma gives $\widehat{\delta}_i(\widehat{v})\le C|\widehat{v}|_{2,\widehat{K}}$.

Suppose that the affine mapping $F_K:\widehat{K}\longrightarrow K$ is defined as

$$\begin{cases} x=x_K+h_{K1}\xi, \\ y=y_K+h_{K2}\eta. \end{cases}$$

On the physical element K, the above process can be expressed as

$$D(v) = GQ(v) + \delta(v), \tag{11}$$

where

$$D(v) = D(\widehat{v} \circ F_K^{-1}), \quad Q(v) = Q(\widehat{v} \circ F_K^{-1}), \quad \delta(v) = \big(\delta_1(v), \delta_2(v), \delta_3(v), \delta_4(v)\big)^{\top},$$

$$\delta_i(v) = \widehat{\delta}_i(\widehat{v}) \circ F_K^{-1} = O(h)|v|_{2,K}, \quad i = 1,2,3,4, \quad G = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}.$$

Dropping $\widehat{\delta}_i(\widehat{v})$ at the right hand of (10), then we get the expressions of β_i of (7) in $Q(\widehat{v})$:

$$\begin{cases} \beta_1 = \frac{1}{4}(\widehat{v}_1 + \widehat{v}_2 + \widehat{v}_3 + \widehat{v}_4), & \beta_2 = \frac{1}{4}(\widehat{v}_2 + \widehat{v}_3 - \widehat{v}_1 - \widehat{v}_4), \\ \beta_3 = \frac{1}{4}(\widehat{v}_3 + \widehat{v}_4 - \widehat{v}_1 - \widehat{v}_2), & \beta_4 = 0. \end{cases} \tag{12}$$

The shape function space on K is defined by

$$P_K = \big\{v = \widehat{v} \circ F_K^{-1}; \ \ \widehat{v} \in \widehat{P} \ \text{ is determined by (7) and (12)}\big\}.$$

Assume $\widehat{I}$ and I_K to be the finite element interpolation operators deduced by $\widehat{P}$ and P_K, respectively, then $I_K v = (\widehat{I}\widehat{v}) \circ F_K^{-1}$, where

$$\widehat{I}\widehat{v} = \beta_1 + \beta_2\xi + \beta_3\eta, \quad \forall\, \widehat{v} \in H^2(\widehat{K}), \tag{13}$$

with the forms of β_1, β_2, β_3 defined as (12).

The finite element space V_h associated with QB element is given by

$$\begin{aligned} V_h = \big\{v_h; \ \ v_h|_K = \widehat{v} \circ F_K^{-1}, \ \widehat{v} \ \text{ is defined by (13)}, \ \ \forall\, K \in \mathcal{J}_h, \\ v(a) = 0, \ \ \text{for any vertex } \ a \ \text{ on } \ \partial\Omega\big\}, \end{aligned} \tag{14}$$

where $\mathcal{J}_h$ is a subdivision of Ω.

Let $I_h : C^0(\Omega) \longrightarrow V_h$ be the global interpolation of QB element on Ω with $I_h|_K = I_K$.

Remark 3.1 It follows from (13) that the interpolation of QB element is the same as the linear part of conforming bilinear element. In fact, one can check that the interpolation of QB element is a disturbance of that of bilinear conforming element, i.e., $(I_K v)(a_i) = v(a_i) + O(h_K)|v|_{2,K}$. So it is a nonconforming element, but this does not influence the good properties of QB element as follows.

Lemma 3.1 (i) Let V_{bh} be the bilinear finite element space, i.e.,

$$V_{bh} = \big\{v \in C^0(\Omega); \ v|_K \in Q_1(K), \ \forall\, K \in \mathcal{J}_h, \ v|_{\partial\Omega} = 0\big\},$$

then we have

$$V_h = \overline{V}_{bh} = \big\{\overline{v}, \ \overline{v} \circ F_K \ \text{ is the linear part of } \ v \circ F_K, \ \ \forall\, K \in \mathcal{J}_h, \ v \in V_{bh}\big\}. \tag{15}$$

(ii) For the interpolation operator I_h, there holds the error estimate

$$\|v - I_h v\|_{0,\Omega} + h\|v - I_h v\|_h \le Ch^2\|v\|_{2,\Omega}, \quad \forall\, v \in H^2(\Omega). \tag{16}$$

4 Error estimates

In this section we consider the nonconforming QB scheme based on the variational formulation (2).

Let $\mathbf{V}_h = V_h \times V_h$, $\mathbf{V}_{bh} = V_{bh} \times V_{bh}$. For $\mathbf{v}_h \in \mathbf{V}_h$, we define $\nabla_h \mathbf{v}_h$ to be the $\mathbf{L}^2(\Omega)$ function whose restriction to each quadrilateral $K \in \mathcal{J}_h$ is given by $\nabla \mathbf{v}_h|_K$. Analogous definitions hold for div_h and rot_h.

Define the discrete semi-norm

$$\|\mathbf{v}_h\|_{m,h} = \Big(\sum_{K \in \mathcal{J}_h} |\mathbf{v}_h|^2_{m,K} \Big)^{\frac{1}{2}}.$$

The nonconforming finite element approximation scheme for problem (2) is then given as follows

$$\begin{cases} \text{Find } \ \mathbf{u}_h \in \mathbf{V}_h, \quad \text{such that} \\ \mu(\nabla_h \mathbf{u}_h, \nabla_h \mathbf{v}_h) + (\lambda + \mu)(\mathrm{div}_h \mathbf{u}_h, \mathrm{div}_h \mathbf{v}_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h. \end{cases} \tag{17}$$

Remark 4.1 Brenner[2] analyzed the nonconforming linear triangular finite element methods for the problem (1). In the nonconforming case, they used the mid-point values on each edge of elements as degrees of freedom, and thus as compared with the finite elements here, the degrees of freedom needed in [2] are about three times of ours.

Remark 4.2 Recently, [4] proposed a nonconforming rectangular element following the ideas of Brenner[2]. It is a P_2 rectangular element with the interpolation operator preserving zero-divergence at each element. Because of the large degrees of freedom, complex construction of this element and nonapplication of more general meshes, it is not an advisable locking free element.

Now we introduce a piecewise constant finite element space

$$M_h = \big\{ q_h \in L_0^2(\Omega);\ q_h|_K \in P_0(K),\ \forall\, K \in \mathcal{J}_h \big\}.$$

As a counterpart of the problem (17), we introduce the following mixed formulation: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$ such that

$$\begin{cases} \mu(\nabla_h \mathbf{u}_h, \nabla_h \mathbf{v}_h) + (\mathrm{div}_h \mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ (\lambda + \mu)^{-1}(p_h, q_h) - (\mathrm{div}_h \mathbf{u}_h, q_h) = 0, \quad \forall\, q_h \in M_h. \end{cases} \tag{18}$$

Then it can be easily proved that the finite element approximation problem (17) is equivalent to the mixed problem (18) with

$$p_h|_K = (\lambda + \mu)\mathrm{div}_h u_h|_K, \quad \forall\, K \in \mathcal{J}_h.$$

Let us stress that the introduction of the new variable is done only as a tool for the mathematical analysis of the method implemented by (17). Based on the above discussion, we can concentrate on the mixed problem (18). In order to establish the uniform convergence with respect to λ, it is sufficient to consider the limiting case when $\lambda \longrightarrow \infty$, which leads to

$$\begin{cases} \mu(\nabla_h \mathbf{u}_h, \nabla_h \mathbf{v}_h) + (\mathrm{div}_h \mathbf{v}_h, p_h) = (\mathbf{f}, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ (\mathrm{div}_h \mathbf{u}_h, q_h) = 0, \quad \forall\, q_h \in M_h. \end{cases} \tag{19}$$

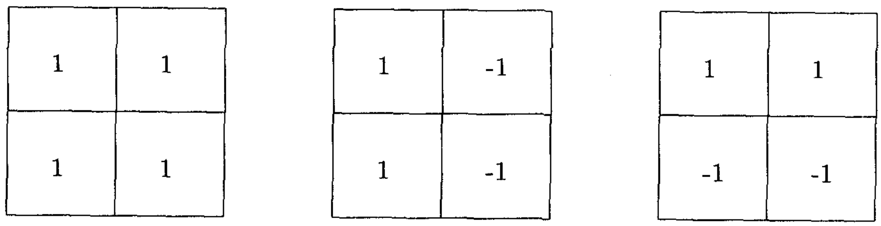
For the sake of convenience, we assume that the subdivision $\mathcal{J}_h$ has been obtained from an arbitrary regular affine quadrilateral meshes $\mathcal{J}_{2h} = \{M\}$ by dividing each element of $\mathcal{J}_{2h}$ into four congruent rectangles. It can be easily seen that the pair $(\mathbf{V}_h, M_h)$ does not always satisfy the inf-sup condition. In fact, for any $\mathbf{v} \in \mathbf{V}_{bh}$, we have $\overline{\mathbf{v}} \in \mathbf{V}_h$, and

$$\int_K q_h \mathrm{div}\overline{\mathbf{v}} \mathrm{d}x\mathrm{d}y = \int_K q_h \mathrm{div}\mathbf{v} \mathrm{d}x\mathrm{d}y, \quad \int_\Omega q_h \mathrm{div}\overline{\mathbf{v}} \mathrm{d}x\mathrm{d}y = \int_\Omega q_h \mathrm{div}\mathbf{v} \mathrm{d}x\mathrm{d}y, \quad \forall\, q_h \in M_h.$$

So the pair $(\mathbf{V}_h, M_h)$ shares the same stable properties of $(\mathbf{V}_{bh}, M_h)$ under affine quadrilateral meshes. It is well-known that the checkerboard function may cause $(\mathbf{V}_{bh}, M_h)$ a spurious pressure mode on a uniform grid. Interested readers can refer to [5-7] for detailed analysis.

We are working with a finite dimensional space $P_h \subset M_h$, such that the pair $(\mathbf{V}_h, P_h)$ satisfies the inf-sup condition. In order to exclude the possibility of spurious or nearly spurious pressure modes, the local basis functions for P_h on a 2×2 patch of M are taken as $\varphi_{i,M}$, which are indicated in Figure 1. Thus we include in P_h all piecewise constants of the form

$$P_h = \Big\{ q_h \in L_0^2(\Omega) : q_h|_M = \sum_{i=1}^{3} \alpha_{i,M}\varphi_{i,M},\ M \in \mathcal{J}_{2h} \Big\}.$$



$\varphi_{1,M}$ $\varphi_{2,M}$ $\varphi_{3,M}$

Figure 1: Local basis functions of P_h

Next, the inf-sup condition for the mixed problem (19) is established.

Lemma 4.1 Under the above hypothesis, the pair $(\mathbf{V}_h, P_h)$ satisfies the following Babuška-Brezzi condition or inf-sup condition[6]

$$\sup_{0 \neq \mathbf{v}_h \in \mathbf{V}_h} \frac{b_h(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,h}} \geq C\|q_h\|_{0,\Omega}, \quad \forall\, q_h \in P_h, \tag{20}$$

where C is a positive constant independent of h.

Proof By the references [5-7], the pair $(\mathbf{V}_{bh}, P_h)$ satisfies a uniform inf-sup condition, i.e.,

$$\sup_{0 \neq \mathbf{v} \in \mathbf{V}_{bh}} \frac{b_h(\mathbf{v}, q_h)}{\|\mathbf{v}\|_{1,\Omega}} \geq C\|q_h\|_{0,\Omega}, \quad \forall\, q_h \in P_h. \tag{21}$$

An equivalent formulation to (21) is: for any $q_h \in P_h$, there is a function $\mathbf{v} \in \mathbf{V}_{bh}$ such that

$$b_h(\mathbf{v}, q_h) \geq C\|q_h\|_{0,\Omega}, \quad \|\mathbf{v}\|_{1,\Omega} \leq C\|q_h\|_{0,\Omega}. \tag{22}$$

Then taking $\overline{\mathbf{v}} \in \mathbf{V}_h$, by the scaling argument and inverse inequality, one can conduct

$$\|\mathbf{v}-\overline{\mathbf{v}}\|_{1,h} \leq Ch\|\mathbf{v}\|_{2,h} \leq C\|\mathbf{v}\|_{1,\Omega}. \tag{23}$$

Thus we have obtained

$$b_h(\overline{\mathbf{v}}, q_h) = b_h(\mathbf{v}, q_h) \geq C\|q_h\|_{0,\Omega}, \tag{24}$$

$$\|\overline{\mathbf{v}}\|_{1,h} \leq C\|\mathbf{v}\|_{1,\Omega} \leq C\|q_h\|_{0,\Omega}, \tag{25}$$

which implies the desired assertion.

Let $\widetilde{p_h} \in P_h$ be the L^2 projection of p_h on P_h. Then the pair $(\mathbf{u}_h, \widetilde{p_h}) \in (\mathbf{V}_h, P_h)$ is the unique solution of the following equations

$$\begin{cases} \mu(\nabla_h \mathbf{u}_h, \nabla_h \mathbf{v}_h) + (\mathrm{div}_h \mathbf{v}_h, \widetilde{p_h}) = (\mathbf{f}, \mathbf{v}_h), \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ (\mathrm{div}_h \mathbf{u}_h, q_h) = 0, \quad \forall\, q_h \in P_h. \end{cases} \tag{26}$$

Now, we are in the position to derive the error estimates.

Theorem 4.1 Let $(\mathbf{u}, p)$ be the solution of problem M_∞ or (4) and $(\mathbf{u}_h, p_h)$ be the solution of problem (18) or (19). Then we have

$$\|\mathbf{u}-\mathbf{u}_h\|_{1,h} + \|p-\widetilde{p_h}\|_{0,\Omega} \leq Ch\|\mathbf{f}\|_{0,\Omega}. \tag{27}$$

Proof Owing to [6], the following error estimate holds

$$\begin{aligned} &\|\mathbf{u}-\mathbf{u}_h\|_{1,h} + \|p-\widetilde{p_h}\|_{0,\Omega} \\ \leq\ & C\Big\{ \inf_{\mathbf{v}_h\in\mathbf{V}_h} \|\mathbf{u}-\mathbf{v}_h\|_{1,h} + \inf_{\widetilde{q_h}\in P_h} \|p-\widetilde{q_h}\|_{0,\Omega} \\ &+ \sup_{\mathbf{v}_h\in\mathbf{V}_h} \frac{\mu(\nabla_h \mathbf{u}, \nabla_h \mathbf{v}_h) + (\mathrm{div}_h \mathbf{v}_h, p) - (\mathbf{f}, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{1,h}} \Big\}. \end{aligned} \tag{28}$$

We only need to bound the three terms at the right hand of (28) one by one.

Suppose $M = \bigcup_{i=1}^4 K_i \in \mathcal{J}_{2h}$ with $K_i \in \mathcal{J}_h\,(1 \leq i \leq 4)$. For the sake of the subsequent analysis, the operator j_h is defined by

$$j_h p = \begin{cases} P_{0K_i} p + \frac{1}{4}\alpha_M, & i = 1, 3, \\ P_{0K_i} p - \frac{1}{4}\alpha_M, & i = 2, 4, \end{cases} \tag{29}$$

where $\alpha_M = P_{0K_2}p + P_{0K_4}p - P_{0K_1}p - P_{0K_3}p$.

A direct calculation shows that $j_h p|_M = \alpha_{1,M}\varphi_{1,M} + \alpha_{2,M}\varphi_{2,M} + \alpha_{3,M}\varphi_{3,M}$, where

$$\alpha_{1,M} = \frac{1}{4}(P_{0K_1}p + P_{0K_2}p + P_{0K_3}p + P_{0K_4}p),$$

$$\alpha_{2,M} = \frac{1}{4}(P_{0K_1}p - P_{0K_2}p - P_{0K_3}p + P_{0K_4}p),$$

$$\alpha_{3,M} = \frac{1}{4}(-P_{0K_1}p - P_{0K_2}p + P_{0K_3}p + P_{0K_4}p).$$

This implies that for any $p \in L_0^2(\Omega)$, $j_h p \in P_h$.

Let $\widehat{\alpha}_{\widehat{M}} = \alpha_M \circ F_K$, it can be checked easily that

$$\widehat{\alpha}_{\widehat{M}} \leq C\|\widehat{p}\|_{0,\widehat{M}} \leq C\|\widehat{p}\|_{1,\widehat{M}}, \quad \widehat{\alpha}_{\widehat{M}} = 0, \quad \forall\, \widehat{p} \in P_0(\widehat{M}). \tag{30}$$

An application of Bramble-Hilbert Lemma yields[13]

$$|\alpha_M| \leq C|\widehat{p}|_{1,\widehat{M}} \leq C|p|_{1,M}, \quad \forall\, p \in H^1(M). \tag{31}$$

Then there holds

$$\begin{aligned}\inf_{\widetilde{q}_h \in P_h} \|p - \widetilde{q}_h\|_{0,\Omega} &\leq \|p - j_h p\|_{0,\Omega} = \Big(\sum_{M \in \mathcal{J}_{2h}} \|p - j_h p\|_{0,M}^2 \Big)^{\frac{1}{2}} \\ &\leq \Big(\sum_{M \in \mathcal{J}_{2h}} \sum_{i=1}^{4} \big(\|p - P_{0K_i} p\|_{0,K_i}^2 + h_K^2 |\alpha_M|^2 \big) \Big)^{\frac{1}{2}} \leq Ch|p|_{1,\Omega}. \end{aligned} \tag{32}$$

Let $\mathbf{I}_h = I_h \times I_h$, and by (16), there holds

$$\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_h \leq \|\mathbf{u} - \mathbf{I}_h \mathbf{u}\|_h \leq Ch\|\mathbf{u}\|_{2,\Omega}. \tag{33}$$

Since

$$\int_F \mathbf{v}_h \mathrm{d}s = \int_F \overline{\mathbf{v}} \mathrm{d}s = \int_F \mathbf{v} \mathrm{d}s$$

is continuous along edges of elements and vanishes if $F \subset \partial\Omega$, where $\mathbf{v} \in \mathbf{V}_{bh}$, $F \subset \partial K$, for all $K \in \mathcal{J}_h$, using the technique developed by [2,9], the consistency error can be estimated as

$$\mu(\nabla_h \mathbf{u}, \nabla_h \mathbf{v}_h) + (\mathrm{div}_h \mathbf{v}_h, p) - (\mathbf{f}, \mathbf{v}_h) \leq Ch\big(\|\mathbf{u}\|_{2,\Omega} + \|p\|_{1,\Omega}\big)\|\mathbf{v}_h\|_{1,h}. \tag{34}$$

A combination of (32)-(34) and the regularity result (3) follows (27), which completes the proof.

In the subsequent part of this section we will show that the nonconforming finite element solution $\mathbf{u}_h$ defined by (17) converges to the solution $\mathbf{u}$ satisfying (2) in energy norm. The energy norm is defined by

$$\|\cdot\|_h = \sqrt{\mu(\nabla_h \cdot, \nabla_h \cdot) + (\lambda + \mu)(\mathrm{div}_h \cdot, \mathrm{div}_h \cdot)}\,.$$

Now, we will turn back to the primitive displacement variable approximation problem (17). The famous second Strang's Lemma[9,13] gives

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C\Big\{ \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_h + \sup_{\mathbf{v}_h \in \mathbf{V}_h \setminus \{0\}} \frac{|E_h(\mathbf{u}, \mathbf{v}_h)|}{\|\mathbf{v}_h\|_h} \Big\}, \tag{35}$$

where

$$E_h(\mathbf{u}, \mathbf{v}_h) = \mu(\nabla_h \mathbf{u}, \nabla_h \mathbf{v}_h) + (\lambda + \mu)(\mathrm{div}_h \mathbf{u}, \mathrm{div}_h \mathbf{v}_h) - (\mathbf{f}, \mathbf{v}_h).$$

Similar to (34), the second term at the right hand of (35), i.e., the consistency error can be estimated as

$$|E_h(\mathbf{u}, \mathbf{v}_h)| \leq Ch\big(\|\mathbf{u}\|_{2,\Omega} + (\lambda + \mu)\|\mathrm{div}\mathbf{u}\|_{1,\Omega}\big)\|\mathbf{v}_h\|_{1,h} \leq Ch\|\mathbf{f}\|_{0,\Omega}\|\mathbf{v}_h\|_h. \tag{36}$$

Let us consider a slight modification of (18): find $(\breve{\mathbf{u}}_h, \breve{p}_h) \in \mathbf{V}_h \times M_h$ such that

$$\begin{cases} \mu(\nabla_h \breve{\mathbf{u}}_h, \nabla_h \mathbf{v}_h) + (\mathrm{div}_h \mathbf{v}_h, \breve{p}_h) = (\mathbf{f}, \mathbf{v}_h), & \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ (\lambda+\mu)^{-1}(p, q_h) - (\mathrm{div}_h \breve{\mathbf{u}}_h, q_h) = 0, & \forall\, q_h \in M_h. \end{cases} \tag{37}$$

Based on the mathematical analysis of problem (19) and (18), one can obtain that

$$\|\mathbf{u} - \breve{\mathbf{u}}_h\|_{1,h} \le Ch\|\mathbf{f}\|_{0,\Omega}, \tag{38}$$

$$\mathrm{div}_h \breve{\mathbf{u}}_h = (\lambda+\mu)^{-1} P_{0h} p = P_{0h} \mathrm{div}\mathbf{u}, \tag{39}$$

where P_{0h} is the L^2 projection operator on M_h.

Then by taking $\mathbf{v}_h = \breve{\mathbf{u}}_h$ in (35), we have

$$\begin{aligned} \|\mathbf{u} - \breve{\mathbf{u}}_h\|_h^2 &= \mu\|\mathbf{u} - \breve{\mathbf{u}}_h\|_{1,h}^2 + (\lambda+\mu)\|\mathrm{div}\mathbf{u} - \mathrm{div}_h \breve{\mathbf{u}}_h\|_{0,\Omega}^2 \\ &\le Ch^2\|\mathbf{f}\|_{0,\Omega}^2 + Ch^2(\lambda+\mu)\|\mathrm{div}\mathbf{u}\|_{1,\Omega}^2 \le Ch^2\|\mathbf{f}\|_{0,\Omega}^2. \end{aligned} \tag{40}$$

Thus we have proved the following important theorem.

Theorem 4.2 Let $\mathbf{u}$, $\mathbf{u}_h$ be the solutions of (2) and (17), respectively, then we have

$$\|\mathbf{u} - \mathbf{u}_h\|_h \le Ch\|\mathbf{f}\|_{0,\Omega}. \tag{41}$$

5 Enhancement of the accuracy

We propose a simple post-processing scheme to improve the accuracy of the finite element solution of problem (17). Here we only state the main result concerning the mathematical analysis. We will present it in our another work.

Noticing that the function $\mathbf{v}_h \in \mathbf{V}_h$ on M can be written as

$$\mathbf{v}_h|_M = \sum_{i=1}^{9} \mathbf{v}_i \phi_i,$$

where $\mathbf{v}_i = \mathbf{v}(a_i)$ and ϕ_i, $i = 1, 2, \cdots, 9$ are the associated basis functions, then we define an interpolation $\Pi_{2h}\mathbf{v}_h \in [Q_2(M)]^2$ by

$$\Pi_{2h}\mathbf{v}_h|_M = \sum_{i=1}^{9} \mathbf{v}_i \psi_i, \quad \forall\, M \in \mathcal{J}_{2h},$$

where ψ_i, $i = 1, 2, \cdots, 9$ are the associated basis functions of the biquadratic element.

Since interpolation $\Pi_{2h}\mathbf{v}_h \in [Q_2(M)]^2$ uses the interpolation theorem, the following theorem is derived directly.

Theorem 5.1 Let $\mathbf{u}$, $\mathbf{u}_h$ be the solution of (2) and (17), respectively, then we have

$$\|\mathbf{u} - \Pi_{2h}\mathbf{u}_h\|_h \le Ch^2\|\mathbf{f}\|_{1,\Omega}.$$

6 Numerical test

We consider the equation (1) with $\mu = 1$, $\mathbf{f} = (f_1, f_2)$ and $\Omega = [0,1] \times [0,1]$, where

$$\begin{cases} f_1 = \pi^2 \big[4\sin 2\pi y(-1+2\cos 2\pi x) - \cos\pi(x+y) + \frac{2}{1+\lambda}\sin\pi x \sin\pi y\big], \\ f_2 = \pi^2 \big[4\sin 2\pi x(-1+2\cos 2\pi y) - \cos\pi(x+y) + \frac{2}{1+\lambda}\sin\pi x \sin\pi y\big]. \end{cases}$$

Then it can be verified that the exact solution of problem (1) is $\mathbf{u} = (u_1, u_2)$ with

$$\begin{cases} u_1 = \sin 2\pi y(-1+2\cos 2\pi x) + \frac{1}{1+\lambda}\sin\pi x \sin\pi y, \\ u_2 = \sin 2\pi x(-1+2\cos 2\pi y) + \frac{1}{1+\lambda}\sin\pi x \sin\pi y. \end{cases}$$

The mesh on Ω is obtained by dividing it into $n \times n$ squares. As comparisons, we computed the energy norm of QB element and the conforming Q_1 element, together with $\|\mathbf{u} - \Pi_{2h}\mathbf{u}_h\|_h$ of QB element. Numerical results are shown in Figure 2 and Figure 3. It is easy to see the robust optimal convergence and superconvergence in the energy norm and the advantage of the QB element over the conforming Q_1 element.

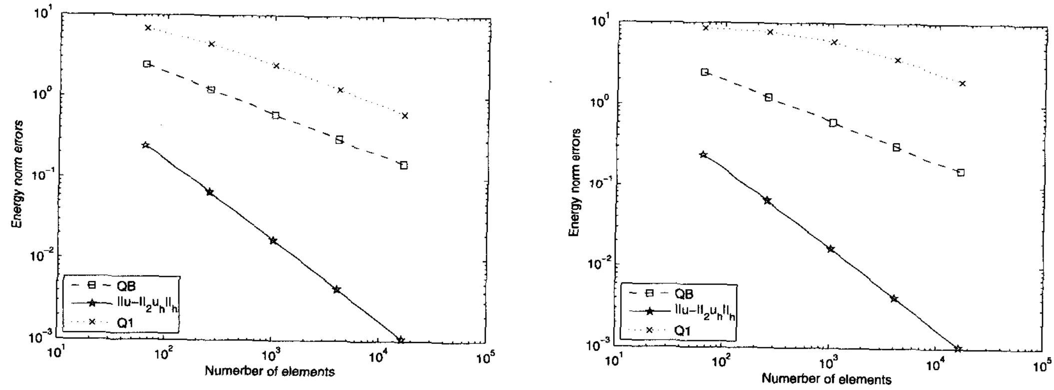


Figure 2: The errors for the case $\lambda = 99$ (left) and $\lambda = 999$ (right)

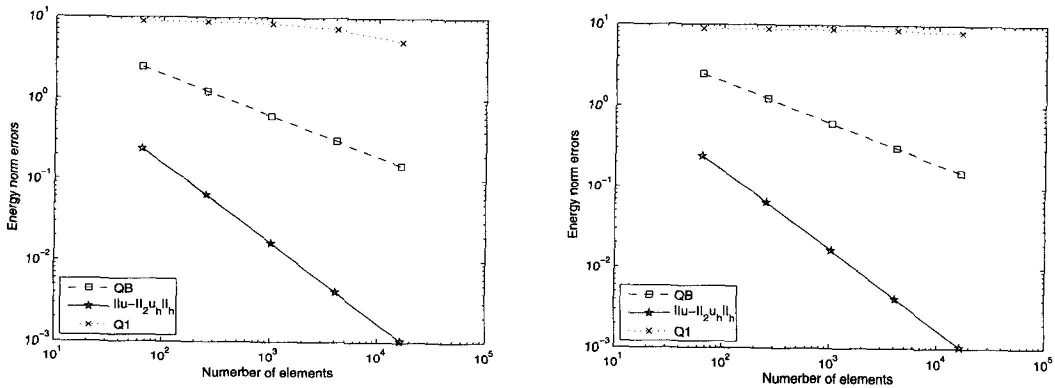


Figure 3: The errors for the case $\lambda = 9999$ (left) and $\lambda = 99999$ (right)

Remark 6.1 The results in this paper can be extend to three dimensional cases with proper modifications.

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线弹性问题的 locking-free 双参数元

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摘 要: 文中运用双参数法提出了一个 4 参数的四边形非协调有限元，讨论了该单元对纯位移边界条件下的均匀介质线弹性方程的逼近问题. 证明了在材料几乎不可压时单元对弹性问题的一致最优收敛性，数值试验验证了理论分析结果. 并通过构造后处理算子，得到了超收敛结果. 所有的分析结果都可以推广到三维情形.

关键词: 四边形非协调元；双参数；无闭锁；线弹性；插值后处理